Exercise Round 3 (December 2, 2013).

Exercise 1. (Wiener velocity model)

Derive the expressions for matrices $A_k$ and $Q_k$ in the exactly discretized version of the model
\begin{equation}
\frac{d^2 x(t)}{dt^2} = w(t),
\end{equation}
where $w(t)$ is a white noise with spectral density $q$, and the sampling period is $t_k - t_{k-1} = \Delta t$.

Exercise 2. (Kalman filter and RTS smoother for O-U model)

Consider the model
\begin{align}
\dot{x} &= -\lambda x dt + d\beta \\
y_k &= x(t_k) + e_k,
\end{align}
with $\lambda = 1/2$, $q = 1$, $x(0) \sim N(0, P_\infty)$, $e_k \sim N(0, 1)$, where $P_\infty$ is the stationary variance of the SDE.

A) Simulate data from the model by using Euler–Maruyama with step size $\Delta t = 1/100$ over the time period $[0, 10]$, and generate measurements at the time steps $t_k = k$ for $k = 1, \ldots, 10$.

B) Implement a Kalman filter to the model with Method A.

C) Implement an RTS smoother (method A) to the problem.

D) How would you compute the smoothing solution at an arbitrary $t$?

Exercise 3. (Continuous-time filtering)

A) Write down the Kushner-Stratonovich equation for the model
\begin{align}
\dot{x} &= -\lambda x dt + d\beta \\
\dot{y} &= x dt + d\eta,
\end{align}
where $\beta$ and $\eta$ are independent standard Brownian motions.

B) Write down the corresponding Zakai equation

C) Write down the Kalman-Bucy filter for the model

D) Show that the filters in A), B), and C) are equivalent
Exercise 4. (GP-regression with O-U covariance function)

A) Implement GP regression for the model

\[ f(t) \sim \mathcal{GP}(0, k(t, t')) \]
\[ y_k = f(t_k) + e_k, \tag{4} \]

where the covariance function is

\[ k(t, t') = \frac{q}{2\lambda} \exp(-\lambda |t - t'|) \tag{5} \]

with the same parameters as in Exercise 2. Apply the GP-regressor to the data simulated in Exercise 2.

B) Compute the spectrum, do the spectral factorization, and form the corresponding state-space representation.

C) Check that the RTS smoother in Exercise 2 gives the same solution as the GP regression above.

Exercise 5. (Approximation of sinusoidal with Matern 3/2)

Recall that the Matern 3/2 covariance function and its spectral density are given as

\[ k(t, t') = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \frac{\sqrt{2\nu} |t - t'|}{l} \right)^\nu K_{\nu} \left( \frac{\sqrt{2\nu} |t - t'|}{l} \right) \]
\[ S(\omega) = \sigma^2 \frac{2\pi^{1/2}\Gamma(\nu + 1/2)}{\Gamma(\nu)} \lambda^{2\nu} (\lambda^2 + \omega^2)^{-(\nu+1/2)} \tag{6} \]

where \( \lambda = \sqrt{2\nu}/l \) and \( \nu = 3/2 \).

A) Generate data from the function \( \sin(t) \) on grid \( \Delta t = 1/100 \) and measurements at times 1, 2, 3, \ldots, 10 with standard deviation 1/10.

B) Implement GP regression with the Matern 3/2 covariance function with \( l = 2, \sigma = 1/2 \), and compute the GP-regression solution at every point on the grid.

C) Derive the state-space representation of the 3/2 Matern process.

D) Implement Kalman filter / RTS smoother solution to the state-space representation of the Matern 3/2 and check that the result matches that of the GP solution in B) above.